Markets with proportional transaction costs and shortsale restrictions

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6th General AMaMeF and Banach Center Conference
June 10-15, 2013

Overview

Model

- $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F}_T = \mathcal{F}$
- risky asset $S = (S_t)_{t=0}^T = (S_t^1, \dots, S_t^d)_{t=0}^T$ - d-dimensional process adapted to $\mathbb{F}$
- risk free asset $B = (B_t)_{t=0}^T$, $B_t \equiv 1$ for all $t = 0, \dots, T$
- trading strategy $H = (H_t)_{t=1}^T = (H_t^1, \dots, H_t^d)_{t=1}^T$-predictable with respect to $\mathbb{F}$
- Let us denote the set of all strategies as $\mathcal{P}$.

Short selling

- Define $\mathcal{P}_+ = \{H \in \mathcal{P} \mid H \geq 0\}$.
- $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\mu = (\mu_1, \ldots, \mu_d)$ where $0 < \lambda_i, \mu_i < 1$
- $\lambda < \mu$ if and only if $\lambda_i < \mu_i$ for $i = 1, \ldots, d$
- Let $\varphi := (\varphi_1, \ldots, \varphi_d)$ where $\varphi^i(x) := x + \lambda_i x^+ + \mu_i x^-$
- Denote

$$(H \cdot S)_t := \sum_{j=1}^{t} H_j \cdot \Delta S_j$$

gain or loss process

GLP is a process $x = (x_t^{\lambda,\mu})_{t=1}^T$ of the form

$$x_t^{\lambda,\mu} := x_t^{\lambda,\mu}(H) = -\sum_{j=1}^{t} \varphi(\Delta H_j) \cdot S_{j-1} - \varphi(-H_t) \cdot S_t =$$

$$= -\sum_{j=1}^{t} \sum_{i=1}^{d} \varphi^i(\Delta H_j^i) S_{j-1}^i - \sum_{i=1}^{d} \varphi^i(-H_t^i) S_t^i$$

where $\Delta H_1^i = H_1^i$.

gain or loss process

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$$= -\sum_{j=1}^{t}\sum_{i=1}^{d} \varphi^i(\Delta H_j^i) S_{j-1}^i - \sum_{i=1}^{d} \varphi^i(-H_t^i) S_t^i$$

where $\Delta H_1^i = H_1^i$. Substituting φ we get

$$x_t^{\lambda,\mu} = (H \cdot S)_t - \sum_{j=1}^{t} \lambda(\Delta H_j)^+ S_{j-1} - \sum_{j=1}^{t} \mu(\Delta H_j)^- S_{j-1} - \mu H_t S_t.$$

(U. Çetin, L.C.G. Rogers, *"Modelling liquidity effects in discrete time"*)

The set of hedgeable claims

Let us define $\mathcal{R}_T^+(\lambda,\mu) := \{x_T^{\lambda,\mu}(H) \mid H \in \mathcal{P}_+\}$ and the set of hedgeable claims as follows

$$\mathcal{A}_T^+(\lambda,\mu) := \mathcal{R}_T^+(\lambda,\mu) - L_+^0.$$

Denote $\overline{\mathcal{A}}_T^+(\lambda,\mu)$ the closure of $\mathcal{A}_T^+(\lambda,\mu)$ in probability.

Remark

$\mathcal{A}_T^+(\lambda,\mu)$ is a convex cone.

absence of arbitrage

Definition (NA_+)

We say that there is no arbitrage in the market if and only if

$$\mathcal{R}_T^+ \cap L_+^0 = \{0\}.$$

(NA_+) is equivalent to the condition $\mathcal{A}_T^+ \cap L_+^0 = \{0\}$.

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(NA_+) is equivalent to the condition $\mathcal{A}_T^+ \cap L_+^0 = \{0\}$. Now we give the definition of robust no arbitrage

Definition (rNA_+)

We say that there is *robust no arbitrage* in the market if and only if

$$\exists\, \varepsilon > 0 \colon (\varepsilon < \lambda,\ \mathcal{A}_T^+(\varepsilon,\mu) \cap L_+^0 = \{0\}) \text{ or } (\varepsilon < \mu,\ \mathcal{A}_T^+(\lambda,\varepsilon) \cap L_+^0 = \{0\}).$$

(W. Schachermayer *"The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time"*)

(λ, μ)-consistent price system

Definition (λ, μ)-CPS

We say that a pair $(\tilde{S}, \mathbb{Q})$ is (λ, μ)-*consistent price system* when $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^{T}$ is an d-dimensional process, adapted to the filtration $\mathbb{F}$ which is $\mathbb{Q}$-martingale and the following inequalities are satisfied

$$1 - \mu_i \leq \frac{\tilde{S}_t^i}{S_t^i} \leq 1 + \lambda_i, \quad \mathbb{P}\text{-a.s.}$$

for all $i = 1, \ldots, d$ and $t = 0, \ldots, T$.

(P. Guasoni, M. Rásonyi, W. Schachermayer *"The fundamental theorem of asset pricing for continuous processes under small transaction costs"*)

(λ, μ)-supermartingale consistent price system

Definition (λ, μ)-supCPS

We say that a pair $(\tilde{S}, \mathbb{Q})$ is (λ, μ)-*supermartingale consistent price system* when $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is an d-dimensional process, adapted to the filtration $\mathbb{F}$ which is $\mathbb{Q}$-supermartingale and the following inequalities are satisfied

$$1 - \mu_i \leq \frac{\tilde{S}_t^i}{S_t^i} \leq 1 + \lambda_i, \quad \mathbb{P}\text{-a.s.}$$

for all $i = 1, \ldots, d$ and $t = 0, \ldots, T$.

right-sided λ-consistent price system

Definition λ-CPS$^+$

We say that a pair $(\tilde{S}, \mathbb{Q})$ is *right-sided λ-consistent price system* when $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^{T}$ is an d-dimensional strictly positive process, adapted to the filtration $\mathbb{F}$ which is $\mathbb{Q}$-martingale and the following inequalities are satisfied

$$\frac{\tilde{S}_t^i}{S_t^i} \leq 1 + \lambda_i, \quad \mathbb{P}\text{-a.s.}$$

for all $i = 1, \ldots, d$ and $t = 0, \ldots, T$.

Necessary conditions for the absence of arbitrage

Main theorem

The implications (a) $\Rightarrow$(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are satisfied where:

(a) $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$ (NA_+);

(b) $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$ and for any $\varepsilon > \lambda$: $\mathcal{A}_T^+(\varepsilon, \mu) = \overline{\mathcal{A}}_T^+(\varepsilon, \mu)$;

(c) for any $\varepsilon > \lambda$: $\overline{\mathcal{A}}_T^+(\varepsilon, \mu) \cap L_+^0 = \{0\}$;

(d) for any $\varepsilon > \lambda$ there exists ε-CPS$^+$ $(\tilde{S}, \mathbb{Q})$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.

Necessary conditions for the absence of arbitrage

Corollary

The implications (a) $\Rightarrow$(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are satisfied where:

(a) $\mathcal{A}_T^+(\lambda,\mu)\cap L^0_+ = \{0\}$; ($NA_+$)

(b) $\mathcal{A}_T^+(\lambda,\mu)\cap L^0_+ = \{0\}$ and for any $\varepsilon>\mu$: $\mathcal{A}_T^+(\lambda,\varepsilon)=\overline{\mathcal{A}}_T^+(\lambda,\varepsilon)$;

(c) for any $\varepsilon>\mu$: $\overline{\mathcal{A}}_T^+(\lambda,\varepsilon)\cap L^0_+ = \{0\}$;

(d) for any $\varepsilon>\mu$ there exists λ-CPS$^+$ $(\tilde{S},\mathbb{Q})$ with $\frac{d\mathbb{Q}}{d\mathbb{P}}\in L^\infty$.

Necessary conditions for the absence of arbitrage

Corollary

The implications (a) $\Rightarrow$(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are satisfied where:

(a) $\mathcal{A}^+_T(\lambda,\mu) \cap L^0_+ = \{0\}$; (NA$_+$)

(b) $\mathcal{A}^+_T(\lambda,\mu) \cap L^0_+ = \{0\}$ and for any $\varepsilon > \mu$: $\mathcal{A}^+_T(\lambda,\varepsilon) = \overline{\mathcal{A}}^+_T(\lambda,\varepsilon)$;

(c) for any $\varepsilon > \mu$: $\overline{\mathcal{A}}^+_T(\lambda,\varepsilon) \cap L^0_+ = \{0\}$;

(d) for any $\varepsilon > \mu$ there exists λ-CPS$^+$ $(\tilde{S}, \mathbb{Q})$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.

Main corollary

(rNA$_+$) $\Rightarrow$ $\exists\, \lambda\text{-CPS}^+$.

Example

- The existence of λ-CPS$^+$ is not a sufficient condition for (NA$_+$).
- Let $T = 2$, $d = 1$, $\lambda = \mu < \frac{1}{3}$ and $S_0 = 1$, $S_1 = 1 + 1\!\!1_A$, $S_2 = \frac{1+\lambda}{1-\lambda}$ where $A \in \mathcal{F}_1$ and $0 < \mathbb{P}(A) < 1$. Furthermore, assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, A, \Omega \setminus A, \Omega\}$.
- Notice that there exists λ-CPS$^+$ in the model. Define $\tilde{S}_t := (1-\mu)E_{\mathbb{Q}}(S_2|\mathcal{F}_t)$ where $\mathbb{Q} \sim \mathbb{P}$ and $t \in \{0,1,2\}$. The measure $\mathbb{Q}$ can be any probability measure equivalent to $\mathbb{P}$ due to the fact that

$$(1-\lambda)E_{\mathbb{Q}}(S_2|\mathcal{F}_1) = (1-\lambda)E_{\mathbb{Q}}(S_2|\mathcal{F}_0) = 1+\lambda.$$

- On the other hand notice that there exists an arbitrage in the model. Define a strategy as follows $\Delta H_1 = H_1 = 1$ and $\Delta H_2 = -1\!\!1_A$. Then

$$x_2^{\lambda,\mu} = -1-\lambda+(2-2\lambda)1\!\!1_A+\left(\frac{1+\lambda}{1-\lambda}-\lambda\frac{1+\lambda}{1-\lambda}\right)1\!\!1_{\Omega\setminus A} = (1-3\lambda)1\!\!1_A.$$

- Finally $\mathcal{A}_2^+(\lambda) \cap L_+^0(\mathcal{F}_2) \neq \{0\}$ despite of existing λ-CPS$^+$.

Sufficient condition for the absence of arbitrage

Theorem

Let the pair $(\tilde{S}, \mathbb{Q})$ will be (λ, μ)-supCPS. Then we have the absence of arbitrage in our model, i.e. $\mathcal{A}_T^{+}(\lambda, \mu) \cap L_+^0 = \{0\}$.

Sufficient condition for the absence of arbitrage

Theorem

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Proof.

Let $\xi \in \mathcal{A}_T^{+}(\lambda, \mu) \cap L_+^0$, i.e. $0 \leq \xi \leq$

$$\leq -\sum_{t=1}^{T} \Delta H_t S_{t-1} + (1-\mu) H_T S_T - \sum_{t=1}^{T} \lambda (\Delta H_t)^{+} S_{t-1} - \sum_{t=1}^{T} \mu (\Delta H_t)^{-} S_{t-1}.$$

We use the inequalities $-\mu_i S_t^i \leq \tilde{S}_t^i - S_t^i \leq \lambda_i S_t^i$, $\mathbb{P}$-a.s. and show that $E_{\mathbb{Q}}(H \cdot \tilde{S})_T \leq 0$. □

Implications

Actually due to the above theorem and the previous example the existence of λ-CPS$^+$ do not imply the existence of (λ, μ)-supCPS.

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Lemma

Assume that the process $(x_t^{\lambda,\mu})_{t=1}^T$ is $\mathbb{Q}$-supermartingale with respect to a measure $\mathbb{Q} \sim \mathbb{P}$. Then there exists a stochastic process $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ such that the pair $(\tilde{S}, \mathbb{Q})$ is λ-CPS$^+$. Moreover, there is no arbitrage in the model, i.e. $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$.

Super-replication

Let C be a contingent claim, i.e. $C \in L^0(\mathcal{F}_T)$. Let us define the set of initial endowments needed to hedge the contingent C.

$$\Gamma^+(C) := \{x \in \mathbb{R} \,|\, \exists\, H \in \mathcal{P}_+ : \; x + x_T^{\lambda,\mu}(H) \geq C, \; \mathbb{P}\text{-a.s.}\}$$

Let

$$\mathcal{Q}^+ := \{\mathbb{Q} \sim \mathbb{P} \,|\, \exists\, \tilde{S} : \; (\tilde{S}, \mathbb{Q}) \text{ is } \lambda\text{-CPS}^+\}.$$

$$\mathcal{Q}_S^+ := \{\mathbb{Q} \sim \mathbb{P} \,|\, \exists\, \tilde{S} : \; (\tilde{S}, \mathbb{Q}) \text{ is } (\lambda, \mu)\text{-supCPS}\}.$$

Super-replication

Let us define also the sets

$$D^+ := \{x \in \mathbb{R} \mid \forall \mathbb{Q} \in \mathcal{Q}^+ : \; E_{\mathbb{Q}} C \leq x\}.$$

$$D_S^+ := \{x \in \mathbb{R} \mid \forall \mathbb{Q} \in \mathcal{Q}_S^+ : \; E_{\mathbb{Q}} C \leq x\}.$$

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Theorem 1

Assume that in the model we have (rNA$_+$).Then $D^+ \subseteq \Gamma^+$.

(Yu. M. Kabanov, M. Rásonyi, Ch. Stricker, *"No-arbitrage criteria for financial markets with efficient friction"*)

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Theorem 2

Assume that there exists $(\lambda, \mu) - supCPS$. Then $\Gamma^+ \subseteq D_S^+$.

Super-replication

Define the super-replication price

$$p_s := \inf \Gamma^+ = \inf\{x \in \mathbb{R} \mid \exists\, H \in \mathcal{P}_+ :\ \ x + x_T^{\lambda,\mu}(H) \geq C,\ \mathbb{P}\text{-a.s.}\}$$

Corollary

Assume that in the model we have (rNA$_+$). Then $p_s \leq \sup_{\mathbb{Q} \in \mathcal{Q}^+} E_{\mathbb{Q}} C$.

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Corollary

Assume that in the model we have (rNA$_+$). Then $p_s \leq \sup_{\mathbb{Q} \in \mathcal{Q}^+} E_{\mathbb{Q}} C$.

Let $\mathcal{Q} := \{\mathbb{Q} \sim \mathbb{P} \mid \exists\, \tilde{S} : \; (\tilde{S}, \mathbb{Q}) \text{ is } (\lambda,\mu)\text{-CPS}\}$.

Corollary

Assume that there exists (λ,μ)-CPS in the model. Then we have the following inequalities

$$\sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}} C \leq \sup_{\mathbb{Q} \in \mathcal{Q}_S^+} E_{\mathbb{Q}} C \leq p_s \leq \sup_{\mathbb{Q} \in \mathcal{Q}^+} E_{\mathbb{Q}} C.$$

Sketch of the proof

In the proof we use the following theorems.

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Stricker's lemma

Let X_n be a sequence of random vectors taking values in $\mathbb{R}^d$ such that for almost all $\omega \in \Omega$ we have $\liminf \|X_n(\omega)\|_d < \infty$. Then there exists a sequence of random vectors Y_n taking values in $\mathbb{R}^d$ such that $Y_n(\omega)$ is a convergent subsequence of $X_n(\omega)$ for almost all $\omega \in \Omega$.

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Kreps-Yan theorem

Let $K \supseteq -L^1_+$ be a closed convex cone in L^1 such that $K \cap L^1_+ = \{0\}$. Then there is a probability $\widetilde{P} \sim P$ with $d\widetilde{P}/dP \in L^\infty$ such that $E_{\tilde{P}}\xi \leq 0$ for all $\xi \in K$.

(Yu. M. Kabanov, C. Stricker, *"A teacher's note on no arbitrage criteria"*)

Sketch of the proof (a) $\Rightarrow$ (b)

(a) $\Rightarrow$ (b) Let

$$x^{\lambda,\mu}_{t,t+\delta}(H,\tilde{H}) = \sum_{j=t}^{t+\delta} H_j \Delta S_j - \sum_{j=t}^{t+\delta} \lambda(\Delta H_j)^+ S_{j-1} - \sum_{j=t}^{t+\delta} \mu(\Delta H_j)^- S_{j-1} - \mu H_{t+\delta} S_{t+\delta}$$

where $1 \leq t \leq t+\delta \leq T$, H is predictable and $H \geq 0$, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and $\Delta H_t = H_t - \tilde{H}$. Define the set

$$\mathcal{R}^+_{t,t+\delta}(\tilde{H},\lambda) := \{x^{\lambda,\mu}_{t,t+\delta}(H,\tilde{H}) \mid H \text{ is predictable and } H \geq 0\}$$

and let $\mathcal{A}^+_{t,t+\delta}(\tilde{H},\lambda) := \mathcal{R}^+_{t,t+\delta}(\tilde{H},\lambda) - L^0_+(\mathcal{F}_{t+\delta})$. We show that the set $\mathcal{A}^+_{t,t+\delta}(\tilde{H},\varepsilon)$ is closed for any $\varepsilon > \lambda$, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and t, δ such that $1 \leq t \leq t+\delta \leq T$. Notice that $\mathcal{A}^+_{1,T}(0,\varepsilon) = \mathcal{A}^+_T(\varepsilon)$.

Sketch of the proof (a) $\Rightarrow$ (b)

Let $\delta = 0$. Fix t, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and vector $\varepsilon > \lambda$. It holds the condition $\mathcal{A}^+_t(\varepsilon) \cap L^0_+(\mathcal{F}_t) = \{0\}$. Suppose that $v^n_{t,t} \to \zeta$ in probability where $v^n_{t,t} \in \mathcal{A}^+_{t,t}(\tilde{H}, \varepsilon)$. By the Riesz theorem the sequence $v^n_{t,t}$ contains a subsequence convergent to ζ a.s. Thus, at most restricting to this subsequence we can assume that $v^n_{t,t} \to \zeta$, $\mathbb{P}$-a.s. Assume that $v^n_{t,t}$ is of the form

$$v^n_{t,t} = H^n_t \Delta S_t - \varepsilon(\Delta H^n_t)^+ S_{t-1} - \mu(\Delta H^n_t)^- S_{t-1} - \mu H^n_t S_t - r_n$$

where $\Delta H^n_t = H^n_t - \tilde{H}$ and $H^n_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$, $r_n \in L^0_+(\mathcal{F}_t)$.

Sketch of the proof (a) $\Rightarrow$ (b)

Consider first the situation on the set $\Omega_1 := \{\liminf \| H_t^n \| < \infty\} \in \mathcal{F}_{t-1}$. By the Stricker's lemma there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable random variables τ_n such that $H_t^{\tau_n}$ is convergent a.s. on Ω_1 and for almost all $\omega \in \Omega_1$ the sequence $H_t^{\tau_n(\omega)}(\omega)$ is a convergent subsequence of the sequence $H_t^n(\omega)$. Notice that $H_t^{\tau_n} \in L^0(\mathbb{R}_+^d, \mathcal{F}_{t-1})$ and respectively $r_{\tau_n} \in L_+^0(\mathcal{F}_t)$. Furthermore r_{τ_n} is convergent a.s. on Ω_1. Let $\tilde{H}_t := \lim_{n\to\infty} H_t^{\tau_n}$ and $\tilde{r} := \lim_{n\to\infty} r_{\tau_n}$. Then

$$\zeta = \lim_{n\to\infty} (H_t^n \Delta S_t - \varepsilon(\Delta H_t^n)^+ S_{t-1} - \mu(\Delta H_t^n)^- S_{t-1} - \mu H_t^n S_t - r_n) =$$

$$= \lim_{n\to\infty} (H_t^{\tau_n} \Delta S_t - \varepsilon(\Delta H_t^{\tau_n})^+ S_{t-1} - \mu(\Delta H_t^{\tau_n})^- S_{t-1} - \mu H_t^{\tau_n} S_t - r_{\tau_n})$$

where the above limit equals

$$\tilde{H}_t \Delta S_t - \varepsilon(\tilde{H}_t - \tilde{H})^+ S_{t-1} - \mu(\tilde{H}_t - \tilde{H})^- S_{t-1} - \mu \tilde{H}_t S_t - \tilde{r} \in \mathcal{A}_{t,t}^+(\tilde{H}, \varepsilon).$$

Sketch of the proof (a) $\Rightarrow$ (b)

It's enough to consider the set $\Omega_2 := \{\liminf \| H_t^n \| = \infty\} \in \mathcal{F}_{t-1}$. Suppose that $\mathbb{P}(\Omega_2) > 0$. Define $G_t^n := \frac{H_t^n}{\|H_t^n\|}$, $h_n := \frac{r_n}{\|H_t^n\|}$ and notice that $G_t^n \in L^0(\mathbb{R}_+^d, \mathcal{F}_{t-1})$. We have

$$G_t^n \Delta S_t - \varepsilon(G_t^n - \frac{\tilde{H}}{\| H_t^n \|})^+ S_{t-1} - \mu(G_t^n - \frac{\tilde{H}}{\| H_t^n \|})^- S_{t-1} - \mu G_t^n S_t - h_n \to 0.$$

Similarly as on the set Ω_1 by the Stricker's lemma there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable random variables σ_n such that $G_t^{\sigma_n}$ is convergent a.s. on Ω_2 and for almost all $\omega \in \Omega_2$ the sequence $G_t^{\sigma_n(\omega)}(\omega)$ is a convergent subsequence of the sequence $G_t^n(\omega)$. Let $\tilde{G}_t := \lim_{n\to\infty} G_t^{\sigma_n}$ and $\tilde{h} := \lim_{n\to\infty} h_{\sigma_n}$.

Sketch of the proof (a) $\Rightarrow$ (b)

Having regard to the absence of short selling we get the equalities

$$\tilde{G}_t \Delta S_t - \varepsilon (\tilde{G}_t)^+ S_{t-1} - \mu (\tilde{G}_t)^- S_{t-1} - \mu \tilde{G}_t S_t = \tilde{G}_t \Delta S_t - \varepsilon \tilde{G}_t S_{t-1} - \mu \tilde{G}_t S_t = \tilde{h}$$

where $\tilde{h} \in L^0_+(\mathcal{F}_t)$. From the absence of arbitrage $\tilde{G}_t \Delta S_t - \varepsilon \tilde{G}_t S_{t-1} - \mu \tilde{G}_t S_t = 0$ on Ω_2. Notice that

$$\tilde{G}_t \Delta S_t - \lambda \tilde{G}_t S_{t-1} - \mu \tilde{G}_t S_t \geq \tilde{G}_t \Delta S_t - \varepsilon \tilde{G}_t S_{t-1} - \mu \tilde{G}_t S_t = 0.$$

Using once again the fact that $\mathcal{A}_t^+(\lambda) \cap L^0_+(\mathcal{F}_t) = \{0\}$ we can replace the inequality by the equality. Hence $\sum_{i=1}^{d} (\lambda_i - \varepsilon_i) \tilde{G}_t^i S_{t-1}^i = 0$. Because S_{t-1} is strictly positive we receive that $\tilde{G}_t = 0$, $\mathbb{P}$-a.s. on Ω_2 what contradicts the fact that $\| \tilde{G}_t \| = 1$. It follows from this that $\mathbb{P}(\Omega_2) = 0$.

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Thank you for your attention!